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Increments of Normal Inverse Gaussian Process as Logarithmic Returns of Stock Price

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***Abstract* – Normal inverse Gaussian (NIG) distribution is quite a new distribution introduced in 1997. This is distribution, which describes evolution of NIG process. It appears that in many cases NIG distribution describes log-returns of stock prices with a high accuracy. Unlike normal distribution, it has higher kurtosis, which is necessary to fit many historical returns. This gives the opportunity to construct precise algorithms for hedging risks of options. The aim of the present research is to evaluate how well NIG distribution can reproduce stock price dynamics and to illuminate future fields of application.**

***Keywords* – Normal inverse Gaussian distribution, normal inverse Gaussian process, log-returns, maximum likelihood estimation.**

I. Introduction

There has been a long quest for chasing the best distribution to describe returns of stock price time series. NIG distribution was introduced to finance society in 1997. From that moment on it has been exercised to underlie many characteristics of stock price dynamics. With higher kurtoses than Gaussian distribution NIG is applicable to many stock price time series data. We have taken stock market data used in "Irrational Exuberance" [1]. Out of given time series, spot price historical development is loaded into R. From given spot prices, an empirical distribution is constructed. As an approximate distribution NIG is chosen [2]–[8]. The corresponding NIG parameters are evaluated by applying maximum likelihood estimation algorithm. The obtained parameters are used to generate NIG random numbers and NIG process. NIG random values are used to construct distribution, which corresponds to distribution of returns of historical stock prices. Afterwards, simulated distribution is compared with empirical distribution.

II. Nomenclature

S	spot price of stock
log	logarithm
NIG	normal inverse Gaussian
Cdf	cumulative distribution function
Qdf	quantile density function
L	likelihood function
f	probability density function
MLE	maximum likelihood estimation
L-BFGS-B	limited-memory Broyden–Fletcher–Goldfarb–Shanno algorithm
KS test	Kolmogorov-Smirnov test

III. Equations

A. Log Returns

Logarithmic return at time *t*:

$\mathrm{X(t)} = \log\left(\frac{S(t)}{S(t-1)}\right)$ is chosen to investigate characteristics of stock price dynamics. It will be used to construct empirical histogram. The NIG approximation will be applied.

B. NIG Distribution

Probability density function of normal inverse Gaussian distribution is defined by the following equation:

$$f(x) = \frac{\alpha\delta K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\pi\sqrt{\delta^2+(x-\mu)^2}} e^{\delta\gamma+\beta(x-\mu)} ,$$

where

K_1 – the modified Bessel function of third order and index 1;
μ – a location parameter;
Δ – a scale parameter;
α – a tail heaviness parameter;
β – an asymmetry parameter;
$\gamma = \sqrt{\alpha^2 - \beta^2}$.

NIG distribution has higher kurtosis than normal distribution. It is very critical for many cases to accurately describe stock price dynamics [3], [4].

IV. NIG Process

NIG process $\mathrm{X_{NIG}} = \left\{X_t^{(NIG)}, t \geq 0\right\}$ which starts at 0 and has independent, stationary increments. The entire process is governed by $\mathrm{NIG}(\mu, \delta, \alpha, \beta)$ distribution. The aim is to simulate stock price dynamics. The internal properties for time series are described by NIG parameters [2], [5].

A. Maximum Likelihood Estimation

As the method to determine the parameters MLE method is chosen. When parameters for NIG are evaluated, it is possible to estimate how well NIG process simulates particular stock price dynamics. The backbone of maximum likelihood estimation can be described in the following way:

We have a sample $\{X_1, X_2, \dots X_N\}$ with corresponding probability density function $f(x_i; \theta)$, where θ is a vector of parameters.

The joint probability density function
$f(x_1, \dots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$

The likelihood function
$$L(\theta | x_1, \dots, x_n) = f(x_1, \dots, x_n; \theta) = \\ = \prod_{i=1}^{n} f(x_i; \theta)$$

Maximum likelihood estimator
$\hat{\theta}_{mle} = \arg\max \log L(\theta | \vec{x})$

By applying random sampling

$$\arg\max \log L(\theta | \vec{x}) = \log\left(\prod_{i=1}^{n} f(x_i; \theta)\right) = \\ = \sum_{i=1}^{n} \log f(x_i; \theta) \qquad (1)$$

First order, nonlinear optimization problem

$$\frac{\partial \log L(\hat{\theta}_{mle} | \vec{x})}{\partial \theta} = \begin{pmatrix} \frac{\partial \log L(\hat{\theta}_{mle} | \vec{x})}{\partial \theta_1} \\ \vdots \\ \frac{\partial \log L(\hat{\theta}_{mle} | \vec{x})}{\partial \theta_k} \end{pmatrix} = \\ = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (2)$$

This equation is solved using L-BFGS-B method [9]–[11].

V. Algorithm

A. Steps of the Algorithm (see Fig. 1)

a) read in stock price time series data in R;
b) construct empirical distribution of stock price data;
c) fit NIG distribution to empirical distribution by applying maximum likelihood estimation (NIG parameters obtained);
d) with evaluated NIG parameters corresponding NIG random variables are generated;
e) stock price returns are compared with NIG process returns.

Necessary packages

```
library("GeneralizedHyperbolic")
library("stats4")
```
[12-13]

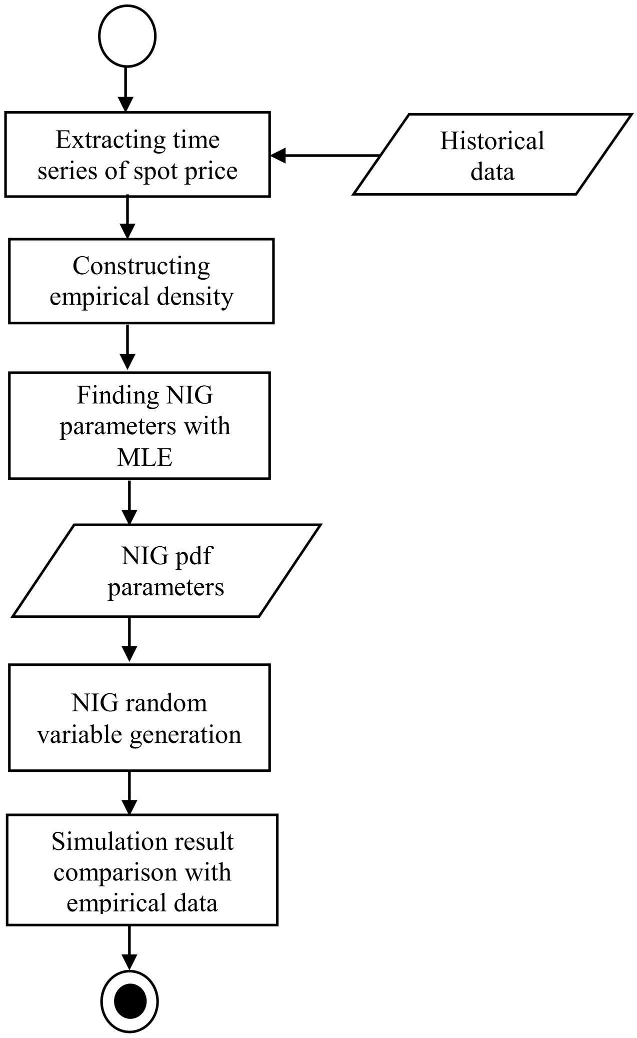

Fig. 1. The flow chart.

VI. Simulation and Results

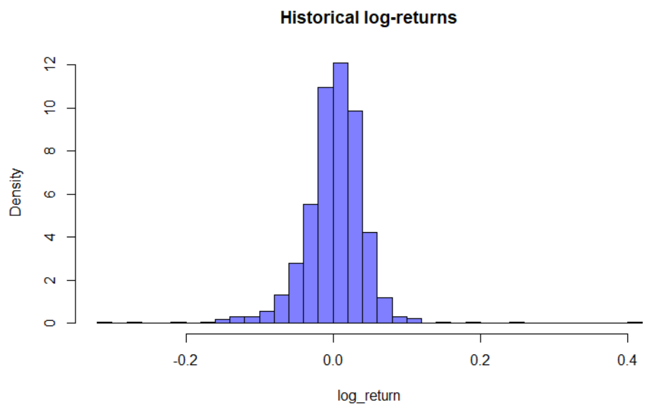

Fig. 2. Empirical probability density function of historical log-returns.

In Fig. 2, empirical probability density function of historical log-returns is visible.

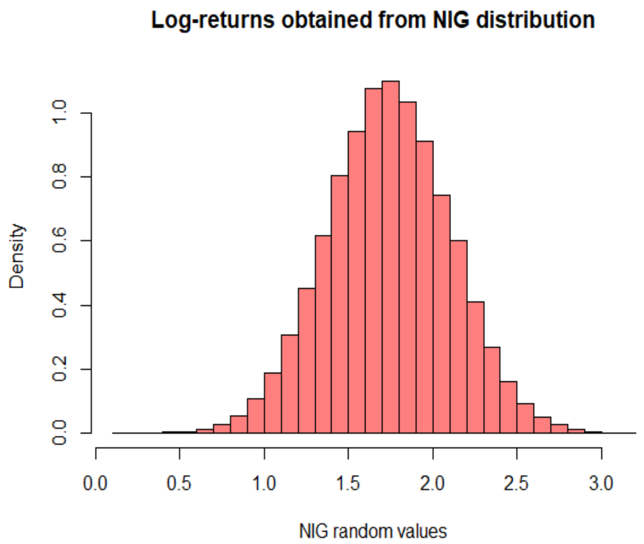

Fig. 3. NIG increments as analogue to log-returns.

NIG distribution is rendered in Fig. 3. To compare correspondence of those 2 distributions, it is necessary to perform two transformations on NIG distribution:

1. to move NIG distribution to left by a mean value and to right by 0.01;
2. to stretch vertically by 11.1.

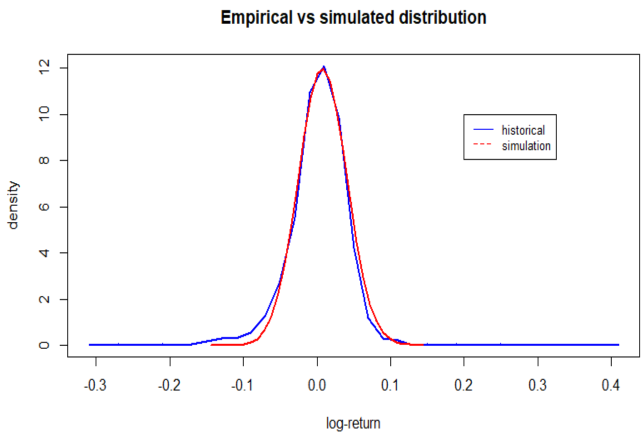

Fig. 4. Empirical vs simulated distribution.

After aforementioned transformations, it is possible to see in Fig. 4 that even from graphical representation it is possible to say that NIG distribution is a good approach to describe particular stock price dynamics. It gives higher kurtosis of empirical logarithmical returns and has lighter tails in comparison to normal distribution. The corresponding NIG distribution is used to construct respective NIG process.

As a numerical closeness measure, a two-sample Kolmogorov-Smirnov test is chosen with the following results obtained (in R):

```
D = 0.029796, p-value = 0.3752
```

Thus, we cannot reject assumption with significance level of 5 % that simulated distribution is good approximation of empirical distribution function. KS test requires sets of random variables as arguments [14]. As the first set we have log-returns but it is necessary to construct the second set from transformed NIG distribution. To achieve it, R spline function is used to obtain a continuous probability density function. The cumulative density function is obtained followed by a quantile function. Quantile function is used in random number generator (in R):

Cumulative distribution function is constructed by means of integration:

```
cdf2<-function(x) integrate(f_spline2,-
0.2,x)$value
```

Afterwards, a quantile function is obtained from cdf:

```
qdf2<-function(x)
optimize(function(z)(cdf(z)-x)^2,c(-
0.2,0.2))$minimum
```

When qdf is obtained, it is easy to build random number generator:

```
rdf2<-function(n) sapply(runif(n),qdf)
```

In Figs. 5–9, the comparison of historical time series and simulated NIG processes is visible. The dynamics of NIG process resembles well dynamics of spot price change. Stock price dynamics is constructed from NIG increments (in R):

First data point is set to be equal with historical spot price:

```
S=price[1]
```

Further, construction of the cumulative sum follows:

```
 for(i in 2:len_dis)
{
 S=append(S,S[i-1]+dis[i]+1)
}
```

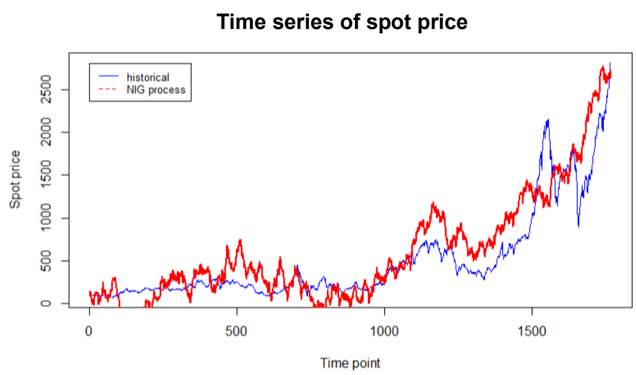

Fig. 5. Simulation of stock price dynamics.

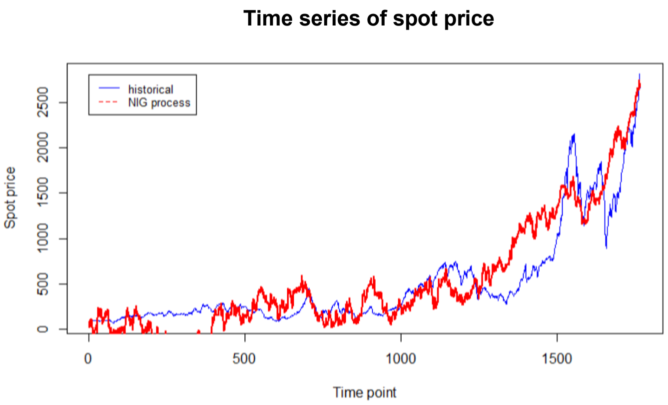

Fig. 6. Simulation of stock price dynamics.

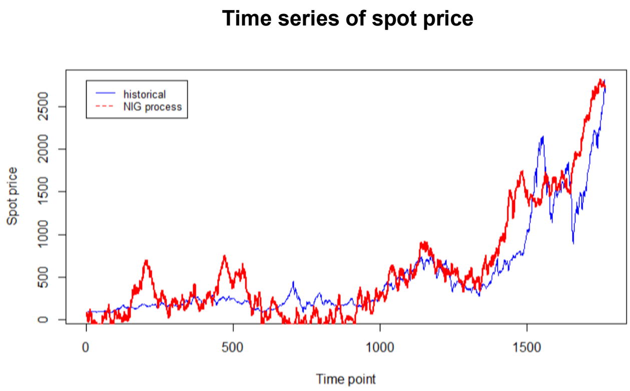

Fig. 7. Simulation of stock price dynamics.

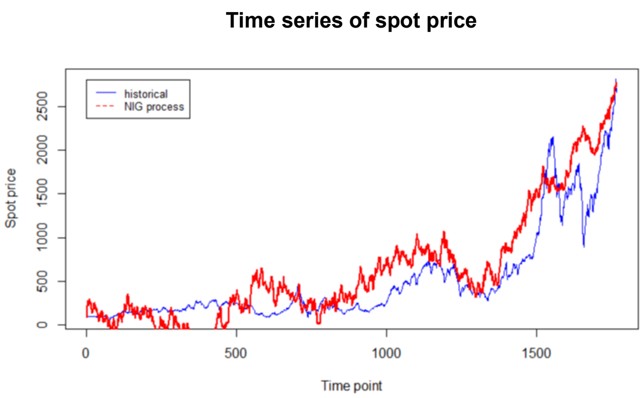

Fig. 8. Simulation of stock price dynamics.

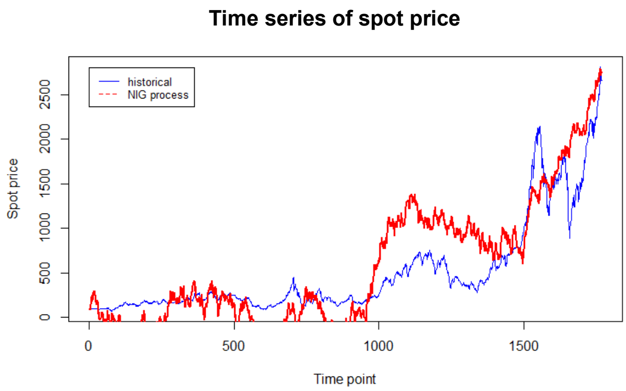

Fig. 9. Initial trajectory.

These trajectories are obtained, after 1000 sets of random NIG values are generated. The one that approximates the historical trajectory the best is chosen. The order of generated random numbers is not changed.

Interesting results are obtained, when a set of random NIG values is permutated (see Figs. 10–12).

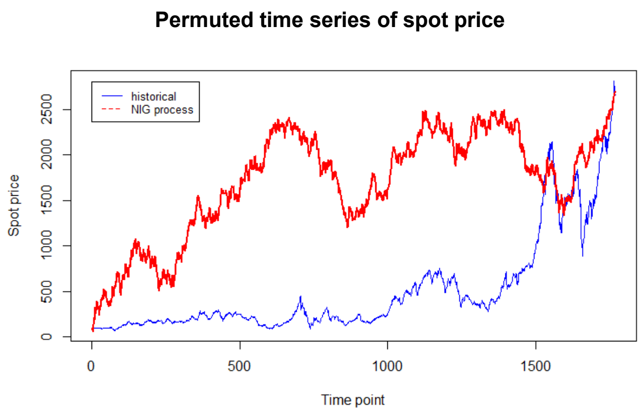

Fig. 10. Permutation 1.

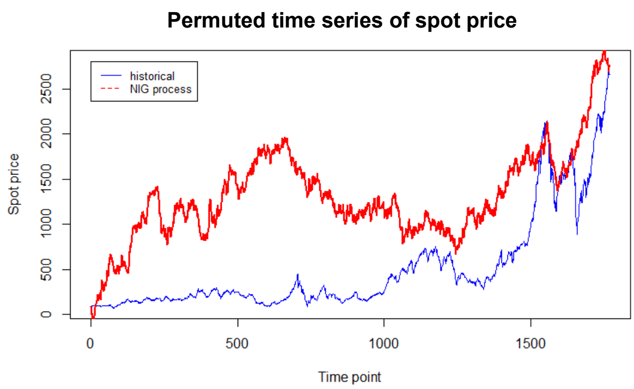

Fig. 11. Permutation 2.

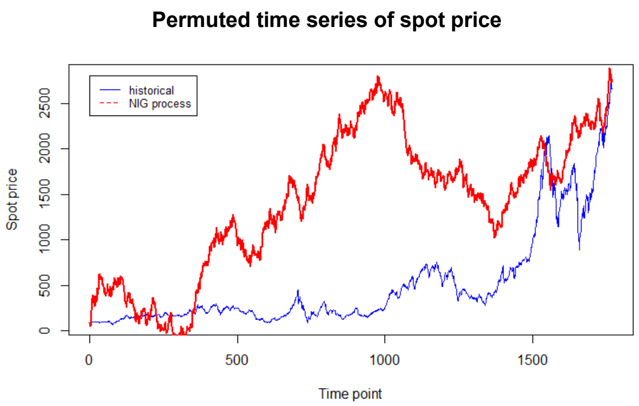

Fig. 12. Permutation 3.

From Figs. 10–12, it is visible that by means of permutations it is possible to achieve that the generated trajectory approximates end teeth of historical time series well. The major part of historical trajectory will be approximated poorly, but the main goal it to show qualitatively that NIG distribution can be used to obtain many qualitative characteristics of stock price dynamics.

VII. Discussion

The high accuracy of NIG approach gives a method to hedge risks for financial derivatives, especially for managing a portfolio of options. As the adjustment of NIG random variables has been used, it is necessary to investigate which transformation of NIG distribution will give the best result in the future analysis. Apart from the mathematical issues, it is also important to find the best way to compare two probability density functions given in tabular format with R.

The authors assume that if order of generated random numbers is changed, the better approximation of stock price dynamics can be achieved. It gives 1769! possible permutations that is an incredibly large number. In the current publication, approximately 100 permutations have been tried, and the best result is rendered in Figs. 10–12. The qualitative goal is achieved: NIG distribution describes the chosen stock price dynamics very well. Unfortunately, it has not been possible to apply NIG distribution without further manual modification.

VIII. Conclusion

Modified normal inverse Gaussian distribution can be used to model log-returns of stock price time series. It is necessary to make further analysis to interpret manual modification of normal inverse Gaussian distribution.

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